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ON SOME INTEGRABLE GENERALISATIONS OF THE CONTINUOUS TODA SYSTEM

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Abstract

In the present paper we obtain some integrable generalisations of the continuous Toda system generated by a flat connection form taking values in higher grading subspaces of the algebra of the area-preserving diffeomorphism of the torus T^2 , and construct their general solutions. The grading condition which we use here, imposed on the connection, can be realised in terms of some holomorphic distributions on the corresponding homogeneous spaces.

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1 Introduction

In the present paper we obtain some integrable generalisations of the continuous Toda system generated by a flat connection form taking values in higher grading subspaces of the Lie Poisson bracket algebra \mathcal{L} on two-dimensional torus T^2 . The main statement is the following. Let \mathcal{M} be a two-dimensional manifold with the local coordinates z_+ and z_- , and G be an exponential mapping of \mathcal{L} defined, e.g., in the sense of inverse limit of Hilbert.⁴ Then,

A flat connection in the trivial fibre bundle $\mathcal{M} \times G \rightarrow \mathcal{M}$ with $(1, 0)$ - and $(0, 1)$ -components taking values in the subspaces $\oplus_{m=0}^{m_+} \mathcal{G}_m$ and $\oplus_{m=0}^{m_-} \mathcal{G}_{-m}$ of the \mathbf{Z} -graded Lie algebra $\mathcal{L} = \oplus_m \mathcal{G}_m$ generates an integrable system of partial differential equations

$$\begin{aligned} \partial_+ \partial_- \rho &= \sum_{m=1}^{m_0} m \frac{\partial^2}{\partial \tau^2} (\omega_m^+ \omega_m^- e^{m\rho}), \quad m_0 \equiv \min(m_+, m_-); \\ \partial_{\mp} \omega_m^{\pm} &= \mp i \sum_{n=1}^{m_{\pm}-m} (m+n) (\omega_{m+n}^{\pm})^{\frac{m}{m+n}} \frac{\partial}{\partial \tau} [\omega_n^{\mp} (\omega_{m+n}^{\pm})^{\frac{n}{m+n}} e^{n\rho}], \end{aligned} \quad (1.1)$$

$1 \leq m \leq m_{\pm} - 1$; for the functions ρ and ω_m^{\pm} depending on three variables z_{\pm} and τ ; $\omega_{m_+}^+ = \omega_{m_-}^- \equiv 1$. The general solution of the Goursat (boundary value) problem to this system is determined by a complete set of arbitrary functions $\Phi_m^+(z_+; \tau)$, $1 \leq m \leq m_+$, and $\Phi_m^-(z_-; \tau)$, $1 \leq m \leq m_-$.

In differential geometry language, the grading condition we use, imposed on the flat connection, can be realised, modifying the reasonings given in [2] for finite-dimensional case, in terms of some holomorphic and antiholomorphic distributions on the homogeneous spaces $F_+ = G/B_-$ and $F_- = G/B_+$, respectively, where B_{\pm} are exponential mappings of $\oplus_{m=0}^{\infty} \mathcal{G}_{\pm m}$. Such a distribution, say on F_+ , is generated by the subspaces of $T^{(1,0)}(F_+)$ with the help of the canonical projection $\pi_+ : G \rightarrow F_+$, as $\pi_{+*}(\mathcal{M}_{+h})$. Here \mathcal{M}_{+h} is obtained by the left translation (with an element h of G , $\pi_+(h) \in F_+$) of the subspace $\mathcal{M}_+ \equiv \oplus_{m \leq m_+} \mathcal{G}_{+m}$ invariant under the adjoint action of B_- in \mathcal{L} . A distribution on F_- is defined similarly, in terms of the canonical projection $\pi_- : G \rightarrow F_-$, and the subspace $\mathcal{M}_- \equiv \oplus_{m \leq m_-} \mathcal{G}_{-m}$.

For the simplest case when the corresponding connection takes values in the local part $\mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{+1}$ of \mathcal{L} , equations (1.1) are reduced to the well-known continuous Toda system (the heavenly equation familiar thanks to [3])

$$\partial_+ \partial_- \rho = \frac{\partial^2}{\partial \tau^2} e^{\rho}; \quad (1.2)$$

for which we briefly discuss the boundary condition on a half-line $z_+ = z_-$. Quite significative, while still rather simply represented case is with $m_{\pm} = 2$, when system (1.1) is reduced to the following equations:

$$\begin{aligned} \partial_+ \partial_- \rho &= \frac{\partial^2}{\partial \tau^2} [2e^{2\rho} + \omega^+ \omega^- e^{\rho}], \\ \partial_{\mp} \omega^{\pm} &= \mp 2i \frac{\partial}{\partial \tau} [\omega^{\mp} e^{\rho}]. \end{aligned} \quad (1.3)$$

⁴Note that an exponential mapping of \mathcal{L} gives a differentiable structure which is weaker than that of a Lie group in the classical sense. Nevertheless, in the same way as for $S_0 \text{ Diff } T^2$, see e.g., [1], it can be defined in the sense of this limit, and just this fact allows us to perform a construction given in the paper.

As a direct by-product of the given consideration one obtains the integrable equations corresponding to the higher grading subspaces of the centreless Virasoro–Witt algebra.

Note that the problem of constructing nonlinear integrable two-dimensional partial differential equations associated with higher \mathbf{Z} -grading subspaces has been already considered in a general and abstract form, in fact, for an arbitrary Lie algebra \mathcal{G} , see [4]. However, even in the simplest case like A_n , the arising equations and their general solutions look, in a sense, more complicated than their continuous limit (1.1).

2 Some Information About Lie Poisson Bracket Algebra

Let us recall briefly some information about Lie Poisson bracket algebra \mathcal{L} following [5], [6]. This algebra, being considered as \mathbf{Z} -graded continuum Lie algebra $\mathcal{G}(E; -i\partial/\partial\tau, -i\partial/\partial\tau) = \bigoplus_{m \in \mathbf{Z}} \mathcal{G}_m$, is isomorphic to A_∞ and $S_0 \text{ Diff } T^2$ (infinitesimal area-preserving diffeomorphisms of the torus T^2). Its elements satisfy the commutation relations

$$[X_m(\phi), X_n(\psi)] = iX_{m+n}(n\dot{\phi}\psi - m\phi\dot{\psi}). \quad (2.1)$$

Here $X_m(\phi) = \int d\tau X_m(\tau) \phi(\tau)$ are the elements of the subspaces \mathcal{G}_m parametrised by the functions $\phi(\tau)$ belonging to the algebra E of trigonometrical polynomials on a circle; $\dot{\phi} \equiv \frac{\partial \phi}{\partial \tau}$.

The algebra $\mathcal{G}(E; -i\frac{\partial}{\partial\tau}; -i\frac{\partial}{\partial\tau})$ is of constant growth (in a functional sense) since $\mathcal{G}_n \simeq \mathcal{G}_1 \simeq E$; its Cartan subalgebra $\wp \simeq \mathcal{G}_0$ is infinite-dimensional; the roots are $n\delta'(\tau)$. Let \wp^* be an algebra dual to \wp , let V be a \mathcal{L} -module and $\lambda \in \wp^*$. Denote by V_λ a set of vectors $v \in V$ satisfying $X_0(\phi)v = \lambda(\phi)v$ for all $\phi \in E$. It can be shown by an appropriate limit procedure (starting from A_r and using the aforementioned isomorphism $\mathcal{G}(E; -i\frac{\partial}{\partial\tau}; -i\frac{\partial}{\partial\tau}) \simeq \mathcal{G}(E; -\frac{\partial^2}{\partial\tau^2}; \text{id}) \simeq A_\infty$) that there exists a nonzero vector $\bar{v} \in V$ such that $\mathcal{G}_m(\bar{v}) = 0$ for $m > 0$ and $U(\mathcal{L})(\bar{v}) = V$. Here $U(\mathcal{L})$ is the universal enveloping algebra for \mathcal{L} . By analogy with the usual (“discrete”) case, this \mathcal{L} -module V is called the highest weight module, and \bar{v} the highest weight vector.

A symmetrical bilinear invariant form on the local part of the algebra in question is defined as follows

$$\text{tr } (X_i(f)X_j(g)) = \delta_{i+j,0}(f, g), \quad i, j = 0, \pm 1;$$

where

$$(f, g) \equiv \int d\tau f(\tau)g(\tau).$$

Thanks to the invariance property of the form,

$$([X_m(f), X_{-m}(g)], X_0(h)) = (X_m(f), [X_{-m}(g), X_0(h)]),$$

the commutation relations (2.1) give

$$-im(X_0(\frac{\partial}{\partial\tau}(fg), X_0(h))) = im(X_m(f), X_{-m}(g\dot{h}));$$

and so one has

$$(X_m(f), X_{-m}(g\dot{h})) = - \int d\tau \frac{\partial}{\partial \tau} (f(\tau)g(\tau))h(\tau) = \int d\tau f g \dot{h}.$$

Thereof,

$$(X_m(f), X_n(g)) = \delta_{m+n,0}(f, g). \quad (2.2)$$

To this end denote a continuous version (in the sense of the algebra A_∞ with the elements $X_m^{(A)}$) of the highest weight vectors of the fundamental representations of A_n by $|\tau\rangle$, for which

$$X_0^{(A)}(\phi) |\tau\rangle = \phi(\tau) |\tau\rangle, \quad X_m^{(A)}(\phi) |\tau\rangle = 0 \text{ for } m \geq 1; \quad (2.3)$$

so that for the case of \mathcal{L} one has

$$X_0(\phi) |\tau\rangle = \partial_\tau^{-1} \phi(\tau) |\tau\rangle, \quad X_m(\phi) |\tau\rangle = 0 \text{ for } m \geq 1. \quad (2.4)$$

3 Derivation of the Equations

Let \mathcal{M} be a two-dimensional manifold with the local coordinates z_+ and z_- in \mathcal{M} , and \mathcal{A} be a Lie algebra \mathcal{G} -valued 1-form on \mathcal{M} , $\mathcal{A} = \mathcal{A}_+ dz_+ + \mathcal{A}_- dz_-$. Any such 1-form generates a connection form of some connection in the trivial fibre bundle. Suppose that \mathcal{A} is flat, so that \mathcal{A}_\pm are some mappings from \mathcal{M} to \mathcal{G} , satisfying the zero curvature condition

$$[\frac{\partial}{\partial z_+} + \mathcal{A}_+, \frac{\partial}{\partial z_-} + \mathcal{A}_-] = 0. \quad (3.1)$$

Let the connection \mathcal{A} takes values in the algebra \mathcal{L} . Choosing the basis $X_m(\phi)$ in $\mathcal{G}(E; -i\frac{\partial}{\partial \tau}; -i\frac{\partial}{\partial \tau})$ and considering the components of the decomposition of \mathcal{A}_\pm over this basis as fields, we can treat (3.1) as a nonlinear system of partial differential equations for the fields. To provide nontriviality of such a system, we impose the grading condition on the connection [4], such that the $(1,0)$ -component \mathcal{A}_+ of \mathcal{A} takes values in $\oplus_{m \geq 0} \mathcal{G}_{+m}$, and the $(0,1)$ -component \mathcal{A}_- takes values in $\oplus_{m \geq 0} \mathcal{G}_{-m}$. Namely,

$$\mathcal{A}_+ = \sum_{m=0}^{m_+} X_m(f_m^+), \quad \mathcal{A}_- = \sum_{m=0}^{m_-} X_{-m}(f_m^-). \quad (3.2)$$

Then substituting (3.2) in (3.1), one has

$$X_0\{\partial_+ f_0^- - \partial_- f_0^+ - i \sum_{m=1}^{m_0} m(f_m^+ f_m^-)\} = 0, \quad (3.3)$$

$$\begin{aligned} \sum_{m=1}^{m_+} X_m\{\partial_- f_m^+ + i m f_m^+ \dot{f}_0^- + i \sum_{1 \leq n < m \leq m_+} X_{m-n}(n \dot{f}_m^+ f_n^- + m f_m^+ \dot{f}_n^-)\} &= 0, \\ \sum_{m=1}^{m_-} X_{-m}\{\partial_+ f_m^- - i m f_m^- \dot{f}_0^+ - i \sum_{1 \leq n < m \leq m_-} X_{-m+n}(n \dot{f}_m^- f_n^+ + m f_m^- \dot{f}_n^+)\} &= 0; \end{aligned}$$

or, in terms of the functions $f^\pm(z_+, z_-)$,

$$\partial_+ f_0^- - \partial_- f_0^+ = i \sum_{m=1}^{m_0} m \frac{\partial}{\partial \tau} (f_m^+ f_m^-) = 0; \quad (3.4)$$

$$\begin{aligned} \partial_- f_m^+ + im f_m^+ \dot{f}_0^- &= -i \sum_{n=1}^{m_+-m} [n \dot{f}_{m+n}^+ f_n^- + (m+n) f_{m+n}^+ \dot{f}_n^-], \quad 1 \leq m \leq m_+; \\ \partial_+ f_m^- - im f_m^- \dot{f}_0^+ &= i \sum_{n=1}^{m_--m} [n \dot{f}_{m+n}^- f_n^+ + (m+n) f_{m+n}^- \dot{f}_n^+]. \quad 1 \leq m \leq m_-. \end{aligned}$$

It follows from the last two systems with $m = m_+$ and $m = m_-$, respectively, that

$$\dot{f}_0^- = \frac{i}{m_+} \partial_- \ln f_{m_+}^+, \quad \dot{f}_0^+ = -\frac{i}{m_-} \partial_+ \ln f_{m_-}^-; \quad (3.5)$$

and from the other equations of these systems

$$\begin{aligned} f_m^+ \partial_- \ln [f_m^+ (f_{m_+}^+)^{-\frac{m}{m_+}}] &= \\ -i \sum_{n=1}^{m_+-m} f_{m+n}^+ f_n^- \frac{\partial}{\partial \tau} \ln [(f_{m_+}^+)^n (f_n^-)^{m+n}], \quad 1 \leq m \leq m_+ - 1; \end{aligned} \quad (3.6)$$

$$\begin{aligned} f_m^- \partial_+ \ln [f_m^- (f_{m_-}^-)^{-\frac{m}{m_-}}] &= \\ i \sum_{n=1}^{m_--m} f_{m+n}^- f_n^+ \frac{\partial}{\partial \tau} \ln [(f_{m_-}^-)^n (f_n^+)^{m+n}], \quad 1 \leq m \leq m_- - 1. \end{aligned} \quad (3.7)$$

Substituting (3.5) in the first equation (3.4) differentiated over τ , one has

$$\partial_+ \partial_- \ln [(f_{m_+}^+)^{1/m_+} (f_{m_-}^-)^{1/m_-}] = \sum_{m=1}^{m_0} m \frac{\partial^2}{\partial \tau^2} f_m^+ f_m^-. \quad (3.8)$$

Then, with the notations

$$f_m^\pm (f_{m_\pm}^\pm)^{-\frac{m}{m_\pm}} \equiv \omega_m^\pm, \quad 1 \leq m \leq m_\pm - 1; \quad (f_{m_+}^+)^{1/m_+} (f_{m_-}^-)^{1/m_-} \equiv e^\rho, \quad (3.9)$$

our equations are finally written as (1.1). In the simplest case when $m_\pm = 1$, this system is reduced to equation (1.2), whose general solution is determined [7] by two arbitrary functions $\Phi^+(z_+, \tau)$ and $\Phi^-(z_-, \tau)$; for $m_\pm = 2$ – to equations (1.3).

The corresponding connection components in terms of the functions ρ and ω_m^\pm are rewritten as

$$\begin{aligned} \mathcal{A}_\pm &= X_0(\mp \frac{i}{m_\mp} \partial_\pm \partial_\tau^{-1} \ln f_{m_\mp}^\mp) \\ &+ \sum_{m=1}^{m_\pm-1} X_{\pm m}(\omega_m^\pm (f_{m_\pm}^\pm)^{\frac{m}{m_\pm}}) + X_{\pm m_\pm}(f_{m_\pm}^\pm); \end{aligned} \quad (3.10)$$

or, in an appropriate gauge, as

$$\mathcal{A}_\pm = X_0(\mp \frac{i}{2} \partial_\pm \partial_\tau^{-1} \rho) + \sum_{m=1}^{m_\pm-1} X_{\pm m}(\omega_m^\pm e^{\frac{m}{2}\rho}) + X_{\pm m_\pm}(e^{\frac{m_\pm}{2}\rho}). \quad (3.11)$$

For the case when all the functions depend on τ and $z_+ - z_-$ only, putting $-i\partial_+ + i\partial_- = \frac{1}{2}\frac{\partial}{\partial r}$ and choosing a gauge with $f_{m\pm}^\pm = e^\rho$, for e.g. $m_\pm = 2$ we have the Lax operator

$$\begin{aligned} L &\equiv \mathcal{A}_+ + \mathcal{A}_- = X_0(\partial_r \partial_\tau^{-1} \rho) \\ &+ X_1(\omega^+ e^{\rho/2}) + X_{-1}(\omega^- e^{\rho/2}) + X_2(e^\rho) + X_{-2}(e^\rho). \end{aligned} \quad (3.12)$$

Then one can calculate, using (2.2), all conserved quantities as $\mathcal{I}_p \equiv \frac{1}{p} \text{tr } L^p$, for which $\frac{\partial}{\partial r} \mathcal{I}_p = 0$, e.g.

$$\mathcal{I}_2 = \int d\tau \left[\frac{1}{2} \left(\frac{\partial y}{\partial r} \right)^2 + e^{\frac{\partial y}{\partial \tau}} \left(e^{\frac{\partial y}{\partial \tau}} + \omega^+ \omega^- \right) \right], \quad (3.13)$$

where $y = \partial_\tau^{-1} \rho$.

Finishing the section, let us note that equations (1.1), remaining integrable, admit the natural reduction to a system associated with the higher grading subspaces of the centreless Virasoro–Witt algebra \mathcal{W} . Indeed, it is clear that with a choice $\phi(\tau) \simeq \tau$, the elements $X_m(\phi)$ of the Lie Poisson bracket algebra \mathcal{L} given by the commutation relations (2.1), generate the algebra \mathcal{W} with the elements L_m satisfying the relations

$$[L_m, L_n] = (m - n) L_{m+n}. \quad (3.14)$$

Then equations (1.1) take the form

$$\begin{aligned} \partial_+ \partial_- \rho &= 2 \sum_{m=1}^{m_0} m \omega_m^+ \omega_m^- e^{m\rho}; \\ \partial_\mp \omega_m^\pm &= \mp i \sum_{n=1}^{m_\pm - m} (m + 2n) \omega_n^\mp \omega_{m+n}^\pm e^{n\rho}, \quad 1 \leq m \leq m_\pm - 1; \end{aligned} \quad (3.15)$$

which for the special case $m_+ = m_- = 2$ coincides with those in [8].

4 Construction of the General Solution

Let us construct the general solution to system (1.1) using an appropriate modification of the method given in [4]. The connection \mathcal{A} is represented in the gradient form,

$$\mathcal{A}_\pm = g^{-1} \partial_\pm g, \quad (4.1)$$

with $g \in G$; and we take for \mathcal{A}_\pm the Gauss type decomposition of g ,⁵

$$g = M_- N_+ g_{0-} \text{ and } g = M_+ N_- g_{0+},$$

respectively. The grading conditions (3.2) provide the holomorphic property of M_\pm , namely that the functions $M_\pm(z_\pm) \in G_\pm$ satisfy the initial value problem

$$\partial_\pm M_\pm(z_\pm) = M_\pm(z_\pm) L_\pm(z_\pm), \quad (4.2)$$

where

$$L_\pm(z_\pm) = \sum_{m=1}^{m_\pm} X_{\pm m}(\Phi_m^\pm(z_\pm)) \quad (4.3)$$

⁵It seems believable that such a decomposition for G can be considered as an appropriate continuous limit of the Gauss decomposition for A_r .

with arbitrary functions $\Phi_m^\pm(z_\pm; \tau)$ determining the general solution to our system. Then one has

$$\mathcal{A}_\pm = g_{0\mp}^{-1}(N_\pm^{-1}\partial_\pm N_\pm)g_{0\mp} + g_{0\mp}^{-1}\partial_\pm g_{0\mp}. \quad (4.4)$$

Differentiating the identity

$$M_+^{-1}M_- = N_-g_0N_+^{-1}, \quad (4.5)$$

with $g_0 \equiv g_{0+}g_{0-}^{-1}$, over z_\pm and using equations (4.2), one can get convinced that the elements $N_\pm^{-1}\partial_\pm N_\pm$ take values in the subspaces $\oplus_{m=1}^{m_\pm} \mathcal{G}_{\pm m}$. Rewrite now decomposition (4.4) as

$$\mathcal{A}_\pm = g_{0\pm}^{-1}\tilde{L}_\pm g_{0\pm} + g_{0\mp}^{-1}\partial_\pm g_{0\mp}, \quad (4.6)$$

where

$$\tilde{L}_\pm \equiv g_0^{\pm 1}(N_\pm^{-1}\partial_\pm N_\pm)g_0^{\mp 1} = \sum_{m=1}^{m_\pm} X_{\pm m}(F_m^\pm)$$

with some functions $F_m^\pm(z_+, z_-)$. In the same way as for the case of a simple Lie algebra, here the functions \tilde{L}_\pm are equal to L_\pm only when $m_+ = m_- = 1$. However, thanks to identity (4.5), the elements N_\pm and g_0 are determined by the elements M_\pm , and hence the functions F_m^\pm entering \tilde{L}_\pm can be expressed in terms of the functions $\Phi_m^\pm(z_\pm)$ for arbitrary values of m_\pm ; moreover, it is clear that $F_{m_\pm}^\pm = \Phi_{m_\pm}^\pm$. A construction of the general solution to system (1.1) can be done in an explicit way. Before we give it, let us discuss briefly the solution to equations (4.2).

The functions $M_\pm(z_\pm) \in G_\pm$ satisfying the initial value problem (4.2) are given by the corresponding multiplicative integrals

$$M_\pm = \sum_{n=0}^{\infty} \int_{Z_\pm^n(z_\pm)} dy_\pm L_\pm(y_\pm^n) L_\pm(y_\pm^{n-1}) \cdots L_\pm(y_\pm^1), \quad (4.7)$$

or, equivalently,

$$M_\pm^{-1} = \sum_{n=0}^{\infty} (-1)^n \int_{Z_\pm^n(z_\pm)} dy_\pm L_\pm(y_\pm^1) L_\pm(y_\pm^2) \cdots L_\pm(y_\pm^n), \quad (4.8)$$

where $Z_\pm^n(z_\pm) = \{y_\pm \in \mathbf{R}^n : a_\pm \leq y_\pm^n \leq y_\pm^{n-1} \leq \cdots \leq y_\pm^1 \leq z_\pm\}$; a_\pm are some constants determining the problem; $M_\pm(a_\pm) = 1$. Note that there is also a noncommutative version of the well-known exponential formula, see [9], [4],

$$\begin{aligned} M_\pm &= \exp \sum_{n=1}^{\infty} \sum_{\omega} \frac{(-1)^{\epsilon(\omega)}}{n^2 C_{\epsilon(\omega)}^{n-1}} \times \\ &\int_{Z_\pm^n} dy_\pm [\cdots [L_\pm(y_\pm^{\omega(n)}), L_\pm(y_\pm^{\omega(n-1)})] \cdots] L_\pm(y_\pm^{\omega(1)}). \end{aligned} \quad (4.9)$$

Here ω is a permutation of the set $\{1, 2, \dots, n\}$; $\epsilon(\omega)$ is the number of errors in the ordering consecutive terms in $\{\omega(1), \omega(2), \dots, \omega(n)\}$; $C_{\epsilon(\omega)}^{n-1}$ are the binomial coefficients.

The algebra $\mathcal{G}(E; -i\partial/\partial\tau, -i\partial/\partial\tau)$, which we have introduced as a \mathbf{Z} -graded algebra, can be represented as \mathbf{Z}^2 -graded algebra with one-dimensional components, e.g., with the elements $X_{\mathbf{m}}$ satisfying the commutation relations [10]

$$[X_{\mathbf{m}}, X_{\mathbf{n}}] = (\mathbf{m} \times \mathbf{n}) X_{\mathbf{m}+\mathbf{n}}, \quad (4.10)$$

where $\mathbf{m} = (m_1, m_2)$ and $\mathbf{n} = (n_1, n_2)$ are two-dimensional integer vectors; $(\mathbf{m} \times \mathbf{n}) \equiv m_1 n_2 - m_2 n_1$. With account of this fact, the multiple commutators of the functions

$$L(y^{(k)}) \equiv \sum_{m_1^{(k)}=1}^{m_{\pm}} X_{m_1^{(k)}}(\Phi_{m_1^{(k)}}^{(k)})$$

in the exponential (4.9) are given by the formula

$$\begin{aligned} & [\cdots [L(y^{(n)}), L(y^{(n-1)})] \cdots] L(y^{(1)}) = \\ & \sum_{m_2^{(1)}, \dots, m_2^{(n)}} \Phi_{\mathbf{m}^{(n)}}^{(n)} \prod_{l=1}^{n-1} ((\sum_{j=l+1}^n \mathbf{m}^{(j)}) \times \mathbf{m}^{(l)}) \Phi_{\mathbf{m}^{(l)}}^{(l)} X_{\sum_{j=1}^n \mathbf{m}^{(j)}}, \end{aligned} \quad (4.11)$$

where

$$\Phi_{m_1^{(k)}}^{(k)}(z; \tau) = \sum_{m_2^{(k)}} \Phi_{\mathbf{m}^{(k)}}^{(k)}(z) e^{im_2^{(k)} \tau}.$$

Now let us give an explicit solution to our system (1.1). Equating expressions (3.11) and (4.6), we have

$$g_{0\mp}^{-1} \partial_{\pm} g_{0\mp} = X_0(\mp \frac{i}{2} \partial_{\pm} \partial_{\tau}^{-1} \rho), \quad (4.12)$$

thereof

$$g_{0\mp} = g_0^{\mp}(z_{\mp}) e^{\mp \frac{i}{2} X_0(\partial_{\tau}^{-1} \rho)},$$

and hence

$$g_0 = g_0^+(z_+) e^{iX_0(\partial_{\tau}^{-1} \rho)} (g_0^-(z_-))^{-1}.$$

Here $g_0^{\pm} \equiv g_0^{\pm}(z_{\pm}) \in G_0$ are arbitrary functions of their arguments, and are expressed in terms of Φ_m^{\pm} . Moreover,

$$\begin{aligned} g_{0\pm}^{-1} \tilde{L}_{\pm} g_{0\pm} &= e^{\mp \frac{i}{2} X_0(\partial_{\tau}^{-1} \rho)} (g_0^{\pm})^{-1} \sum_{m=1}^{m_{\pm}} X_{\pm m}(F_m^{\pm}) g_0^{\pm} e^{\pm \frac{i}{2} X_0(\partial_{\tau}^{-1} \rho)} \\ &= \sum_{m=1}^{m_{\pm}-1} X_{\pm m}(\omega_m^{\pm} e^{\frac{m}{2} \rho}) + X_{\pm m_{\pm}}(e^{\frac{m_{\pm}}{2} \rho}). \end{aligned} \quad (4.13)$$

Rewriting identity (4.5) in the form

$$(g_0^+)^{-1} M_+^{-1} M_- g_0^- = [(g_0^+)^{-1} N_- g_0^+] e^{iX_0(\partial_{\tau}^{-1} \rho)} [(g_0^-)^{-1} N_+^{-1} g_0^-],$$

one has

$$< \tau | (g_0^+)^{-1} M_+^{-1} M_- g_0^- | \tau > = < \tau | e^{iX_0(\partial_{\tau}^{-1} \rho)} | \tau >, \quad (4.14)$$

where the brackets are taken between the basis vector $|\tau\rangle$ and its dual, $\langle \tau|$, annihilated by the subspaces \mathcal{G}_+ and \mathcal{G}_- , respectively. This matrix element realises a continuous version of the tau-function depending on the necessary number of arbitrary functions $\Phi_m^{\pm}(z_{\pm}, \tau)$ which determine the general solution to system (1.1). It can be rewritten as a series over the nested integrals of the products of these functions in the same way as it was done for the case of the continuous Toda system (1.2) in terms of the basis (2.4) with the help of the commutation relations (2.1), see e.g., [6], or with the discrete basis (4.10) and formulae (4.11).

Now, since, in accordance with (2.1),

$$e^{X_0(\Phi)} X_m(\Psi) e^{-X_0(\Phi)} = X_m(e^{im\partial_\tau \Phi} \Psi), \quad (4.15)$$

one has from (4.13) that

$$(g_0^\pm)^{-1} \sum_{m=1}^{m_\pm} X_{\pm m}(F_m^\pm e^{\frac{m}{2}\rho}) g_0^\pm = \sum_{m=1}^{m_\pm-1} X_{\pm m}(\omega_m^\pm e^{\frac{m}{2}\rho}) + X_{\pm m_\pm}(e^{\frac{m_\pm}{2}\rho});$$

and hence

$$\sum_{m=1}^{m_\pm} X_{\pm m}(F_m^\pm) = \sum_{m=1}^{m_\pm-1} g_0^\pm X_{\pm m}(\omega_m^\pm) (g_0^\pm)^{-1} + g_0^\pm X_{\pm m_\pm} (g_0^\pm)^{-1}. \quad (4.16)$$

Thanks to (4.16), with the parametrisation $g_0^\pm = e^{X_0(\partial_\tau^{-1}\nu_\pm)}$, one has $F_{m_\pm}^\pm = \Phi_{m_\pm}^\pm = e^{\pm im_\pm \nu_\pm}$, so that

$$g_0^\pm(z_\pm) = e^{\mp \frac{i}{m_\pm} X_0(\partial_\tau^{-1} \ln \Phi_{m_\pm}^\pm(z_\pm))}. \quad (4.17)$$

Now expression (4.14) implies, with account of (2.4) and (4.17), that

$$e^\rho = (\Phi_{m_+}^+)^{\frac{1}{m_+}} (\Phi_{m_-}^-)^{\frac{1}{m_-}} e^{-i\partial_\tau^2 \ln \langle \tau | M_+^{-1} M_- | \tau \rangle}, \quad (4.18)$$

where $M_\pm(z_\pm)$ are determined from the initial value problem (4.2) by the infinite series (4.7), (4.8), or exponentials (4.9). Here the continuous analogue of the tau-function $\langle \tau | M_+^{-1} M_- | \tau \rangle$ is represented by the series

$$\begin{aligned} & \sum_{m,n=0}^{\infty} (-1)^n \int_{Z_+^n(z_+)} dy_+ \int_{Z_-^m(z_-)} dy_- \times \\ & \langle \tau | L_+(y_+^1) \cdots L_+(y_+^n) L_-(y_-^m) L_-(y_-^1) | \tau \rangle, \end{aligned} \quad (4.19)$$

where nonzero contributions come only from the terms with the same number of the elements L_+ and L_- in their products in (4.19), i.e.,

$$\begin{aligned} \langle \tau | M_+^{-1} M_- | \tau \rangle &= 1 + \sum_{n \geq 1} (-1)^n \int_{Z_+^n(z_+)} dy_+ \int_{Z_-^n(z_-)} dy_- \times \\ & \langle \tau | L_+(y_+^1) \cdots L_+(y_+^n) L_-(y_-^n) L_-(y_-^1) | \tau \rangle. \end{aligned} \quad (4.20)$$

The rule for explicit calculating the matrix elements in series (4.20) consists in successive displacements of all L_+ and L_- to the extreme right and, respectively, left position at which they annihilate the highest vectors $|\tau\rangle$ and $\langle \tau|$. In this one uses the decomposition of L_\pm as series (4.3), and relations (2.4) and (2.1).

Moreover, from (4.16) we obtain

$$X_{\pm m}(F_m^\pm) = g_0^\pm X_{\pm m}(\omega_m^\pm) (g_0^\pm)^{-1} = X_{\pm m}(\omega_m^\pm e^{\frac{m}{m_\pm} \ln \Phi_{m_\pm}^\pm}),$$

$1 \leq m \leq m_\pm - 1$, and hence

$$\omega_m^\pm = F_m^\pm (\Phi_{m_\pm}^\pm)^{-\frac{m}{m_\pm}}, \quad 1 \leq m \leq m_\pm - 1. \quad (4.21)$$

So, formulas (4.18) and (4.21) define the general solution to system (1.1).

The functions F_m^\pm , $1 \leq m \leq m_\pm - 1$, entering solution (4.21), and in turn \tilde{L}_\pm , are determined in terms of the matrix elements of the known element $M_+^{-1}M_-$ taken between some, not necessarily highest vectors of the representation space. Consider, for example, the matrix element with the highest bra-vector $\langle \tau |$ and a ket-vector $|\tau \rangle^{(1)}$ which is annihilated by the action of the subspaces \mathcal{G}_m , $m > 1$. Then

$$\langle \tau | g_0^{-1} M_+^{-1} M_- | \tau \rangle^{(1)} = \langle \tau | (g_0^{-1} N_- g_0) N_+^{-1} | \tau \rangle^{(1)} = \langle \tau | N_+^{-1} | \tau \rangle^{(1)},$$

and differentiating this equality over z_+ , one has a sequence of equalities

$$\begin{aligned} \partial_+ \langle \tau | g_0^{-1} M_+^{-1} M_- | \tau \rangle^{(1)} &= - \langle \tau | g_0^{-1} \tilde{L}_+ g_0 N_+^{-1} | \tau \rangle^{(1)} = \\ &= - \langle \tau | g_0^{-1} X_1(F_1^+) g_0 N_+^{-1} | \tau \rangle^{(1)} = - \langle \tau | g_0^{-1} X_1(F_1^+) g_0 | \tau \rangle^{(1)}, \end{aligned}$$

which determines the function F_1^+ . Now, knowing this function, to find F_2^+ we consider the matrix element with a ket-vector $|\tau \rangle^{(2)}$ which is annihilated by the action of the subspaces \mathcal{G}_m , $m > 2$; etc. The similar procedure allows to determine the functions F_m^- .

It seems interesting to investigate the problem of the integrability of system (1.1) on a half-line $r = 0$. Let us study it by the simplest example of the heavenly equation (1.2) which shows that this problem respects the boundary condition

$$\partial_r x = 0 \quad \text{at } r = 0; \quad \rho \equiv \partial_\tau^2 x. \quad (4.22)$$

To get convinced in it one can equate the corresponding W_m^\pm -elements, $\partial_\mp W_m^\pm = 0$, at $z_+ = z_-$. Then it follows from the expression

$$W_2^\pm = \int d\tau [\partial_\pm^2 x + \frac{1}{2}(\partial_\pm \partial_\tau x)^2],$$

that

$$\partial_r x = \lambda \cdot e^{\rho/2} \quad \text{at } r = 0, \quad (4.23)$$

where λ is an arbitrary constant yet; formally the similar condition takes place for the case of the Liouville equation [11]. Then, already from the W^\pm -elements of the 3rd order,

$$W_3^\pm = \int d\tau [\tau \partial_\pm^3 x + (\partial_\pm x) \cdot (\partial_\pm^2 \partial_\tau x) + \tau (\partial_\pm \partial_\tau x) \cdot (\partial_\pm^2 \partial_\tau x) - \frac{1}{3}(\partial_\pm \partial_\tau x)^3],$$

one sees that this constant λ should be equal to zero. Recall that for the affine Toda field theory associated with the series $A_n^{(1)}$, as well as it can be easily shown for the corresponding finite case A_n , there are two possibilities for constants λ_i entering the boundary conditions like (4.23), or its affine deformation, written for the fields x_i , $1 \leq i \leq r$; namely $|\lambda_i| = 2$, or all $\lambda_i = 0$ [12].⁶ It is interesting to note that nonabelian Toda systems, in general, admit not such a rigid restriction on these constants. In particular, for the case of B_2 , the boundary conditions contain one arbitrary constant, as it can be seen by equating the corresponding W^\pm -elements, all three of them being of the 2nd order [13].

⁶The same as for the Liouville case, the sine (or sinh)–Gordon model is less restricted in this sense; see [12] for the relevant references.

5 Algebraic Structure of the Solution

The matrix element (4.20) determining the general solution to system (1.1) realises a continuous version of the well-known object in the theory of integrable systems – the tau-function. At the same time this matrix element is closely related to the so-called Shapovalov form defined here on the Lie algebra \mathcal{L} , that can clarify an algebraic structure of the solution. Such a relationship is quite common and takes place for a wide class of nonlinear integrable systems, including, in particular, abelian and nonabelian Toda systems associated with the simple Lie algebras \mathcal{G} ; let us discuss it in brief.

Recall, see e.g. [14], that the standard Shapovalov form defines the linear mapping $U(\mathcal{G}) \otimes_{\mathbf{C}} U(\mathcal{G}) \mapsto U(\wp)$ and is realised as a bilinear form $(x^\vee y)_0$ for any two elements $x, y \in U(\mathcal{G})$. Here $x \rightarrow x^\vee$ is the Chevalley involution for $x^\vee = x'$, and the hermitean Chevalley involution for $x^\vee = x^*$; the subscript 0 means the projection of $U(\mathcal{G})$ on $U(\wp)$ which is parallel to $\mathcal{G}_- U(\mathcal{G}) + U(\mathcal{G}) \mathcal{G}_+$. The given definition is naturally extended for the case of the algebra $U'(\mathcal{G}) = U(\mathcal{G}) \otimes_{U(\wp)} R(\wp^*)$, where $R(\wp^*)$ is the algebra of the rational functions over \wp^* . It is very important to note that the form $(x^\vee y)_0$ is degenerated on the left ideal $U'(\mathcal{G}) \mathcal{G}_+$, and is not degenerated on the subalgebra $U(\mathcal{G}_-)$; and hence is not degenerated on the space $U'(\mathcal{G})/U'(\mathcal{G}) \mathcal{G}_+$ which is a rational span of the corresponding Verma module. In fact, those forms which are most relevant for describing the general solution to the Toda type equations [4], are related with an appropriate holomorphic extension of the algebra $U(\mathcal{G})$, namely $U_h(\mathcal{G}) = U(\mathcal{G}) \otimes_{U(\wp)} h(\wp^*)$ where $h(\wp^*)$ is the algebra of holomorphic functions over \wp^* . Note that a special class of such extensions was introduced in [15], see also [16]. In our case we deal with a holomorphic extension of the algebra $U(\mathcal{L})$, and the role of the holomorphic and antiholomorphic functions is played by the functions $\Phi_m^+(z)$ and $\Phi_m^-(\bar{z})$, respectively, under a relevant reality condition imposed, similarly to those in [2] for the Toda system, on the solution to equations (1.1).

A basis in $U(\mathcal{L})$ is constructed with the help of the monomials $\hat{X}(\{M\}) = X_{m_n} \cdots X_{m_1}$ in terms of the basis elements $X_m = X_m(\Phi_m)$ of \mathcal{L} , see (4.19). In particular, let $\hat{X}_\pm(\{M\})$, $\hat{X}_-(\{M\}) = \hat{X}_+^\vee(\{M\})$, is such a basis in $U(\mathcal{G}_\pm)$ with the weight $\mu(\{M\})$. Then the elements $\hat{X}_+(\{M\}) \hat{X}_-(\{N\})$ generate a basis of $U'(\mathcal{L})$ over $R(\wp^*)$, and this procedure gives for any weight $\mu \in \wp^*$ a vector space $F_\mu(\mathcal{L})$ of all formal series $\sum_{\{M, N\}} c_{\{MN\}} \hat{X}_+(\{M\}) \hat{X}_-(\{N\})$ with $c_{\{MN\}} \in R(\wp^*)$, where the sum runs over all monomials of the weight $\mu = \mu(\{M\}) - \mu(\{N\})$. The subspaces $F_\mu(\mathcal{L})$ are in turn the subspaces of the algebra $F(\mathcal{L})$ graded by the weights μ . To calculate explicitly the corresponding Shapovalov form or its hermitean version, $(x^\vee y)$, one needs to transform the elements $\hat{X}_+(\{M\}) \hat{X}_-(\{N\})$ entering (4.20) to the series of the monomials $\hat{X}_- \hat{X}_0 \hat{X}_+$, with $\hat{X}_0 \in U(\wp)$, by using the commutation relations (2.1) of the algebra \mathcal{L} . This is just the rule that we have mentioned below formula (4.20). Now one can get convinced that the matrix element (4.20) is a Shapovalov type form $(x^\vee y)_0$ for some special two elements $x, y \in U_h(\mathcal{G}_-)$ of the Lie algebra \mathcal{L} in question.

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